PRICING SYNTHETIC CDO TRANCHES IN A MODEL WITH DEFAULT CONTAGION USING THE MATRIX-ANALYTIC APPROACH

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Abstract. We value synthetic CDO tranche spreads, index CDS spreads, k^{th}-to-default swap spreads and tranchelets in an intensity-based credit risk model with default contagion. The default dependence is modelled by letting individual intensities jump when other defaults occur. The model is reinterpreted as a Markov jump process. This allows us to use a matrix-analytic approach to derive computationally tractable closed-form expressions for the credit derivatives that we want to study. Special attention is given to homogenous portfolios. For a fixed maturity of five years, such a portfolio is calibrated against CDO tranche spreads, index CDS spread and the average CDS spread, all taken from the iTraxx Europe series. After the calibration, which renders perfect fits, we compute spreads for tranchelets and k^{th}-to-default swap spreads for different subportfolios of the main portfolio. Studies of the implied tranche-losses and the implied loss distribution in the calibrated portfolios are also performed. We implement two different numerical methods for determining the distribution of the Markov-process. These are applied in separate calibrations in order to verify that the matrix-analytic method is independent of the numerical approach used to find the law of the process. Monte Carlo simulations are also performed to check the correctness of the numerical implementations.

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1. Introduction

In recent years the market for synthetic CDO tranches and index CDS-s, which are derivatives with a payoff linked to the credit loss in a portfolio of CDS-s, have seen a rapid growth and increased liquidity. This has been followed by an intense research for understanding and modelling the main feature driving these products, namely default dependence.

In this paper we derive computationally tractable closed-form expressions for synthetic CDO tranche spreads and index CDS spreads. This is done in an intensity based model where default dependencies among obligors are expressed in an intuitive, direct and compact way. The financial interpretation is that the individual default intensities are constant, except at the times when other defaults occur: then the default intensity for each obligor jumps by an amount representing the influence of the defaulted entity on that obligor. This phenomena is often called default contagion. The above model is then reinterpreted in terms of a Markov jump process. This interpretation makes it possible to use a matrix-analytic approach to derive practical formulas for CDO tranche spreads and index CDS spreads. Our approach is the same as in (Herbertsson 2005) and (Herbertsson & Rootzén 2008) where the authors study aspects of k^{th}-to default spreads in nonsymmetric as well as in symmetric portfolios. The contribution of this paper is a continuation of this technique to synthetic CDO tranches and index CDS-s.

Except for (Herbertsson 2005) and (Herbertsson & Rootzén 2008), the methods presented in (Bielecki, Crépey, Jeanblanc & Rutkowski 2006), (Bielecki, Vidozzi & Vidozzi 2006), (Davis & Esparragoza 2007), (Davis & Lo 2001*a*), (Davis & Lo 2001*b*), (Frey & Backhaus 2004), (Frey & Backhaus 2008), (Backhaus 2008), Section 5.9 in (Lando 2004) and Subsection 9.8.3 in (McNeil, Frey & Embrechts 2005), (Laurent, Cousin & Fermanian 2008), (Cont & Minca 2008), (Arnsdorf & Halperin 2007) are currently closest to the approach of this article. The framework used here (and in (Herbertsson 2005) and (Herbertsson & Rootzén 2008)) is the same as in (Frey & Backhaus 2004), (Frey & Backhaus 2008), (Backhaus 2008) and is related to (Bielecki, Crépey, Jeanblanc & Rutkowski 2006), (Bielecki, Vidozzi & Vidozzi 2006). The main differences are that (Frey & Backhaus 2004), (Frey & Backhaus 2008), (Backhaus 2008) use time-varying parameters in their practical examples and then solve the corresponding Chapman-Kolmogorov equation using numerical methods for ODE-systems. Furthermore, in (Backhaus 2008), the author also consider numerical examples where the portfolio is split into homogeneous groups with default contagion both within each group and between groups. (Bielecki, Vidozzi & Vidozzi 2006) use Monte Carlo simulations to calibrate and price the instruments.

Default contagion in an intensity based setting have previously also been studied in for example (Avellaneda & Wu 2001), (Bielecki & Rutkowski 2001), (Collin-Dufresne, Goldstein & Hugonnier 2004), (Giesecke & Weber 2004), (Giesecke & Weber 2006), (Jarrow & Yu 2001), (Kraft & Steffensen 2007), (Rogge & Schönbucher 2003), (Schönbucher & Schubert 2001) and (Yu 2007). The material in all these papers and books are related to the results discussed here.

This paper is organized as follows. In Section 2 we give an introduction to synthetic CDO tranches and index CDS-s which motivates results and introduces notation needed in the sequel. Section 3 presents the intensity-based model for default contagion. Using a result from (Herbertsson & Rootzén 2008), the model is reinterpreted in terms of a Markov jump process. Section 4 presents convenient analytical formulas for synthetic CDO tranche spreads and index CDS spreads. We assume that the recovery rates are deterministic and that the interest rate is constant. In Section 5 we apply the results from Section 4 to a homogenous model. Then, in Section 6, for a fixed maturity of five years, this portfolio is calibrated against CDO tranche spreads, the index CDS spread and the average CDS spread, all taken from the iTraxx series, resulting in perfect fits. We use three different iTraxx series, sampled before and during the subprime-crises. We also give a careful discussion regarding the numerical methods for determining the distribution of the Markov-process, and their influence on the calibrations as well as other aspects. It is shown that the calibrations are insensitive to the two numerical methods that are used. After the calibration, we compute k^{th}-to-default swap spreads for different subportfolios of the main portfolio. This problem is slightly different from the corresponding one in previous studies, e.g. (Herbertsson 2005) and (Herbertsson & Rootzén 2008), since the obligors undergo default contagion both from the subportfolio and from obligors outside the subportfolio, in the main portfolio. Further, we compute spreads on tranchelets which are nonstandard CDO tranches with smaller loss-intervals than standardized tranches. We also investigate implied tranche-losses and the implied loss distribution in the calibrated portfolios. The final section, Section 7 summarizes and discusses the results and compares our model with similar frameworks in the recent credit derivative literature.

2. Valuation of Synthetic CDO tranche spreads and index CDS spreads

In this section we give a short description of tranche spreads in synthetic CDO-s and of index CDS spreads. It is independent of the underlying model for the default times and introduces notation needed later on.

2.1. **The cash-flows in a synthetic CDO.** In this section and in the sequel all computations are assumed to be made under a risk-neutral martingale measure $\mathbb{P}$. Typically such a $\mathbb{P}$ exists if we rule out arbitrage opportunities. Further, we assume the that risk-free interest rate, r_t is deterministic.

A synthetic CDO is defined for a portfolio consisting of m single-name CDS's on obligors with default times $\tau_1, \tau_2 \ldots, \tau_m$ and recovery rates $\phi_1, \phi_2, \ldots, \phi_m$. It is standard to assume that the nominal values are the same for all obligors, denoted by N. The accumulated credit loss L_t at time t for this portfolio is

$$L_t = \sum_{i=1}^{m} N(1-\phi_i)1_{\{\tau_i \leq t\}}. \tag{2.1.1}$$

We will without loss of generality express the loss L_t in percent of the nominal portfolio value at $t = 0$. For example, if all obligors in the portfolio have the same constant recovery rate ϕ, then $L_{T_k} = k(1-\phi)/m$ where $T_1 < \ldots < T_k$ is the ordering of $\tau_1, \tau_2, \ldots, \tau_m$.

A CDO is specified by the attachment points $0 = k_0 < k_1 < k_2 < \dots k_\kappa = 1$ with corresponding tranches $[k_{\gamma-1}, k_\gamma]$. The financial instrument that constitutes tranche γ with maturity T is a bilateral contract where the protection seller **B** agrees to pay the protection buyer **A**, all losses that occur in the interval $[k_{\gamma-1}, k_\gamma]$ derived from L_t up to time T. The payments are made at the corresponding default times, if they arrive before T, and at T the contract ends. The expected value of this payment is called the *protection leg*, denoted by $V_\gamma(T)$. As compensation for this, **A** pays **B** a periodic fee proportional to the current outstanding (possibly reduced due to losses) value on tranche γ up to time T. The expected value of this payment scheme constitutes the *premium leg* denoted by $W_\gamma(T)$. The accumulated loss $L_t^{(\gamma)}$ of tranche γ at time t is

$$L_t^{(\gamma)} = (L_t - k_{\gamma-1})\, 1_{\{L_t \in [k_{\gamma-1}, k_\gamma]\}} + (k_\gamma - k_{\gamma-1})\, 1_{\{L_t > k_\gamma\}}. \tag{2.1.2}$$

Let $B_t = \exp\left(-\int_0^t r_s ds\right)$ denote the discount factor where r_t is the risk-free interest rate. The protection leg for tranche γ is then given by

$$V_\gamma(T) = \mathbb{E}\left[\int_0^T B_t dL_t^{(\gamma)}\right] = B_T \mathbb{E}\left[L_T^{(\gamma)}\right] + \int_0^T r_t B_t \mathbb{E}\left[L_t^{(\gamma)}\right] dt,$$

where we have used integration by parts for Lebesgue-Stieltjes measures together with Fubini-Tonelli and the fact that r_t is deterministic. Further, if the premiums are paid at $0 < t_1 < t_2 < \ldots < t_{n_T} = T$ and if we ignore the accrued payments at defaults, then the premium leg is given by

$$W_\gamma(T) = S_\gamma(T) \sum_{n=1}^{n_T} B_{t_n} \left(\Delta k_\gamma - \mathbb{E}\left[L_{t_n}^{(\gamma)}\right]\right) \Delta_n$$

where $\Delta_n = t_n - t_{n-1}$ denote the times between payments (measured in fractions of a year) and $\Delta k_\gamma = k_\gamma - k_{\gamma-1}$ is the nominal size of tranche γ (as a fraction of the total nominal value of the portfolio). The constant $S_\gamma(T)$ is called the spread of tranche γ and is determined so that the value of the premium leg equals the value of the corresponding protection leg.

2.2. **The tranche spreads.** By definition, the constant $S_\gamma(T)$ is determined at $t = 0$ so that $V_\gamma(T) = W_\gamma(T)$, that is, so that the value of the premium leg agrees with the corresponding protection leg. Furthermore, for the first tranche, often denoted the *equity* tranche, $S_1(T)$ is set to 500 bp and a so called *up-front* fee $S_1^{(u)}(T)$ is added to the premium leg so that $V_1(T) = S_1^{(u)}(T) k_1 + W_1(T)$. Hence, we get that

$$S_\gamma(T) = \frac{B_T \mathbb{E}\left[L_T^{(\gamma)}\right] + \int_0^T r_t B_t \mathbb{E}\left[L_t^{(\gamma)}\right] dt}{\sum_{n=1}^{n_T} B_{t_n}\left(\Delta k_\gamma - \mathbb{E}\left[L_{t_n}^{(\gamma)}\right]\right)\Delta_n} \qquad \gamma = 2, \ldots, \kappa$$

and

$$S_1^{(u)}(T) = \frac{1}{k_1}\left[B_T \mathbb{E}\left[L_T^{(1)}\right] + \int_0^T r_t B_t \mathbb{E}\left[L_t^{(1)}\right] dt - 0.05 \sum_{n=1}^{n_T} B_{t_n}\left(\Delta k_1 - \mathbb{E}\left[L_{t_n}^{(1)}\right]\right)\Delta_n\right].$$

The spreads $S_\gamma(T)$ are quoted in bp per annum while $S_1^{(u)}(T)$ is quoted in percent per annum. Note that spreads are independent of the nominal size of the portfolio.

2.3. **The index CDS spread.** Consider the same synthetic CDO as above. An index CDS with maturity T, has almost the same structure as a corresponding CDO tranche, but with two main differences. First, the protection is on *all* credit losses that occurs in the CDO portfolio up to time T, so in the protection leg, the tranche loss $L_t^{(\gamma)}$ is replaced by the total loss L_t. Secondly, in the premium leg, the spread is paid on a notional proportional to the number of obligors left in the portfolio at each payment date. Thus, if N_t denotes the number of obligors that have defaulted up to time t, i.e $N_t = \sum_{i=1}^m 1_{\{\tau_i \leq t\}}$, then the index CDS spread $S(T)$ is paid on the notional $(1 - \frac{N_t}{m})$. Since the rest of the contract has the same structure as a CDO tranche, the value of the premium leg $W(T)$ is

$$W(T) = S(T) \sum_{n=1}^{n_T} B_{t_n} \left(1 - \frac{1}{m} \mathbb{E}\left[N_{t_n}\right]\right) \Delta_n$$

and the value of the protection leg, $V(T)$, is given by $V(T) = B_T \mathbb{E}\left[L_T\right] + \int_0^T r_t B_t \mathbb{E}\left[L_t\right] dt$. The index CDS spread $S(T)$ is determined so that $V(T) = W(T)$ which implies

$$S(T) = \frac{B_T \mathbb{E}\left[L_T\right] + \int_0^T r_t B_t \mathbb{E}\left[L_t\right] dt}{\sum_{n=1}^{n_T} B_{t_n} \left(1 - \frac{1}{m} \mathbb{E}\left[N_{t_n}\right]\right) \Delta_n} \tag{2.3.1}$$

where $\frac{1}{m}\mathbb{E}\left[N_t\right] = \frac{1}{1-\phi}\mathbb{E}\left[L_t\right]$ if $\phi_1 = \phi_2 = \ldots = \phi_m = \phi$. The spread $S(T)$ is quoted in bp per annum and is independent of the nominal size of the portfolio.

2.4. **The expected tranche losses.** From Subsection 2.2 we see that to compute tranche spreads we have to compute $\mathbb{E}\left[L_t^{(\gamma)}\right]$, that is, the expected loss of the tranche $[k_{\gamma-1}, k_\gamma]$ at time t. If we let $F_{L_t}(x) = \mathbb{P}\left[L_t \leq x\right]$ then (2.1.2) implies that

$$\mathbb{E}\left[L_t^{(\gamma)}\right] = (k_\gamma - k_{\gamma-1})\, \mathbb{P}\left[L_t > k_\gamma\right] + \int_{k_{\gamma-1}}^{k_\gamma} (x - k_{\gamma-1})\, dF_{L_t}(x). \tag{2.4.1}$$

Hence, in order to compute $\mathbb{E}\left[L_t^{(\gamma)}\right]$ and $\mathbb{E}\left[L_t\right]$ and we must know the loss distribution $F_{L_t}(x)$ at time t. Furthermore, if the recoveries are nonhomogeneous, then to determine the index CDS spread, we also must compute $\mathbb{E}\left[N_{t_n}\right]$, which is equivalent to finding the default distributions $\mathbb{P}\left[\tau_i \leq t\right]$ for all obligors, or alternatively determining the distributions $\mathbb{P}\left[T_k \leq t\right]$ for all ordered default times T_k.

3. Intensity based models reinterpreted as Markov jump processes

In this section we define the intensity-based model for default contagion which is used throughout the paper. The model is then translated into a Markov jump process. This makes it possible to use a matrix-analytic approach to derive computationally convenient formulas for CDO tranche spreads, index CDS spreads, single-name CDS spreads and k^{th}-to-default spreads. The model presented here is identical to the setup in (Herbertsson &

Rootzén 2008) where the authors study aspects of k^{th}-to-default spreads in nonsymmetric as well as in symmetric portfolios. In this paper we focus on synthetic CDO trances, index CDS and k^{th}-to-default swaps on subportfolios to the CDO portfolio.

With $\tau_1, \tau_2 \dots, \tau_m$ default times as above, define the point process $N_{t,i} = 1_{\{\tau_i \leq t\}}$ and introduce the filtrations

$$\mathcal{F}_{t,i} = \sigma\left(N_{s,i}; s \leq t\right), \quad \mathcal{F}_t = \bigvee_{i=1}^{m} \mathcal{F}_{t,i}.$$

Let $\lambda_{t,i}$ be the $\mathcal{F}_t$-intensity of the point processes $N_{t,i}$. Below, we for convenience often omit the filtration and just write intensity or "default intensity". With a further extension of language we will sometimes also write that the default times $\{\tau_i\}$ have intensities $\{\lambda_{t,i}\}$. The model studied in this paper is specified by requiring that the default intensities have the form,

$$\lambda_{t,i} = a_i + \sum_{j \neq i} b_{i,j} 1_{\{\tau_j \leq t\}}, \qquad \tau_i \geq t, \tag{3.1}$$

and $\lambda_{t,i} = 0$ for $t > \tau_i$. Further, $a_i \geq 0$ and $b_{i,j}$ are constants such that $\lambda_{t,i}$ is non-negative.

The financial interpretation of (3.1) is that the default intensities are constant, except at the times when defaults occur: then the default intensity for obligor i jumps by an amount $b_{i,j}$ if it is obligor j which has defaulted. Thus a positive $b_{i,j}$ means that obligor i is put at higher risk by the default of obligor j, while a negative $b_{i,j}$ means that obligor i in fact benefits from the default of j, and finally $b_{i,j} = 0$ if obligor i is unaffected by the default of j.

Equation (3.1) determines the default times through their intensities. However, the expressions for the loss and tranche losses are in terms of their joint distributions. It is by no means obvious how to go from one to the other. Here we will use the following result, proved in (Herbertsson & Rootzén 2008).

Proposition 3.1. *There exists a Markov jump process $(Y_t)_{t\geq 0}$ on a finite state space $\boldsymbol{E}$ and a family of sets $\{\Delta_i\}_{i=1}^m$ such that the stopping times*

$$\tau_i = \inf\left\{t > 0 : Y_t \in \Delta_i\right\}, \qquad i = 1, 2, \dots, m,$$

have intensities (3.1). Hence, any distribution derived from the multivariate stochastic vector $(\tau_1, \tau_2, \dots, \tau_m)$ can be obtained from $\{Y_t\}_{t\geq 0}$.

Each state $\boldsymbol{j}$ in $\boldsymbol{E}$ is of the form $\boldsymbol{j} = \{j_1, \dots j_k\}$ which is a subsequence of $\{1, \dots m\}$ consisting of k integers, where $1 \leq k \leq m$. The interpretation is that on $\{j_1, \dots j_k\}$ the obligors in the set have defaulted. The Markov jump process Y_t on $\boldsymbol{E}$ is specified by making $\{1, \dots m\}$ absorbing and starting in $\{0\}$.

In this paper, Proposition 3.1 is throughout used for computing distributions. However, we still use Equation (3.1) to describe the dependencies in a credit portfolio since it is more compact and intuitive. In the sequel, we let $\boldsymbol{Q}$ and $\boldsymbol{\alpha}$ denote the generator and initial distribution on $\boldsymbol{E}$ for the Markov jump process in Proposition 3.1. The generator $\boldsymbol{Q}$ is found by using the structure of $\boldsymbol{E}$, the definition of the states $\boldsymbol{j}$, and Equation (3.1), see (Herbertsson & Rootzén 2008). By construction $\boldsymbol{\alpha} = (1, 0, \dots, 0)$. Further, if $\boldsymbol{j}$ belongs to $\boldsymbol{E}$ then $\boldsymbol{e_j}$ denotes a column vector in $\mathbb{R}^{|\boldsymbol{E}|}$ where the entry at position $\boldsymbol{j}$ is 1 and the other entries are zero. From Markov theory we know that $\mathbb{P}\left[Y_t = \boldsymbol{j}\right] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t}\boldsymbol{e_j}$ were $e^{\boldsymbol{Q}t}$ is the matrix exponential which has a closed form expression in terms of the eigenvalue decomposition of $\boldsymbol{Q}$.

4. Using the matrix-analytic approach to find CDO tranche spreads and index CDS spreads

In this section we derive practical formulas for CDO tranche spreads and index CDS spreads. This is done under (3.1) together with the standard assumption of deterministic recovery rates and constant interest rate r. Although the derivation is done in an inhomogeneous portfolio, we will in Section 5 show that these formulas are almost the same in a homogeneous model.

The following observation is a key to all results in this article. If the obligors in a portfolio satisfy (3.1) and have deterministic recoveries, then Proposition 3.1 implies that the corresponding loss L_t can be represented as a functional of the Markov jump process Y_t, $L_t = L\left(Y_t\right)$ where the mapping L goes from $\boldsymbol{E}$ to all possible loss-outcomes determined via (2.1.1). For example, if $\boldsymbol{j} \in \boldsymbol{E}$ where $\boldsymbol{j} = \{j_1, \dots j_k\}$ then $L\left(\boldsymbol{j}\right) = \frac{1}{m}\sum_{n=1}^{k}(1-\phi_{j_n})$. The range of L is a finite set since the recoveries are deterministic. This implies that for any mapping $g(x)$ on $\mathbb{R}$ and a set A in $[0,\infty)$, we have

$$\int_A g(x)dF_{L_t}(x) = \boldsymbol{\alpha} e^{\boldsymbol{Q}t}\boldsymbol{h}(g, A)$$

where $\boldsymbol{h}(g,A)$ is a column vector in $\mathbb{R}^{|\boldsymbol{E}|}$ defined by $\boldsymbol{h}(g,A)_{\boldsymbol{j}} = g(L(\boldsymbol{j}))1_{\{L(\boldsymbol{j})\in A\}}$. From this we obtain the following easy lemma, which is stated since it provides notation which is needed later on.

Lemma 4.1. *Consider a synthetic CDO on a portfolio with m obligors that satisfy (3.1). Then, with notation as above,*

$$\mathbb{E}\left[L_t^{(\gamma)}\right] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t}\boldsymbol{\ell}^{(\gamma)}\ , \quad \mathbb{E}\left[L_t\right] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t}\boldsymbol{\ell} \quad \textit{and} \quad \mathbb{E}\left[N_t\right] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t}\sum_{i=1}^{m}\boldsymbol{h}^{(i)}$$

where $\boldsymbol{\ell}^{(\gamma)}$ is a column vector in $\mathbb{R}^{|\boldsymbol{E}|}$ defined by

$$\boldsymbol{\ell}_{\boldsymbol{j}}^{(\gamma)} = \begin{cases} 0 & \textit{if} \quad L(\boldsymbol{j}) < k_{\gamma-1} \\ L(\boldsymbol{j}) - k_{\gamma-1} & \textit{if} \quad L(\boldsymbol{j}) \in [k_{\gamma-1}, k_\gamma] \\ \Delta k_\gamma & \textit{if} \quad L(\boldsymbol{j}) > k_\gamma \end{cases} \tag{4.1}$$

and L is the mapping such that $L_t = L(Y_t)$. Furthermore, $\boldsymbol{\ell}$ and $\boldsymbol{h}^{(i)}$ are column vectors in $\mathbb{R}^{|\boldsymbol{E}|}$ defined by $\boldsymbol{\ell}_{\boldsymbol{j}} = L(\boldsymbol{j})$ and $\boldsymbol{h}_{\boldsymbol{j}}^{(i)} = 1_{\{\boldsymbol{j}\in\Delta_i\}}$ where the sets Δ_i are as in Proposition 3.1.

We now present the following convenient formulas. Proofs are given in Appendix.

Proposition 4.2. *Consider a synthetic CDO on a portfolio with m obligors that satisfy (3.1) and assume that the interest rate r is constant. Then, with notation as above,*

$$S_\gamma(T) = \frac{\left(\boldsymbol{\alpha} e^{\boldsymbol{Q}T} e^{-rT} + \boldsymbol{\alpha}\boldsymbol{R}(0,T)r\right)\boldsymbol{\ell}^{(\gamma)}}{\sum_{n=1}^{n_T} e^{-rt_n}\left(\Delta k_\gamma - \boldsymbol{\alpha} e^{\boldsymbol{Q}t_n}\boldsymbol{\ell}^{(\gamma)}\right)\Delta_n} \qquad \gamma = 2,\dots,\kappa \tag{4.2}$$

and

$$S_1^{(u)}(T) = \frac{1}{k_1}\left(\boldsymbol{\alpha} e^{\boldsymbol{Q}T} e^{-rT} + \boldsymbol{\alpha}\boldsymbol{R}(0,T)r + 0.05\sum_{n=1}^{n_T}\boldsymbol{\alpha} e^{\boldsymbol{Q}t_n} e^{-rt_n}\Delta_n\right)\boldsymbol{\ell}^{(1)} - 0.05\sum_{n=1}^{n_T} e^{-rt_n}\Delta_n \tag{4.3}$$

where

$$\boldsymbol{R}(0,T) = \int_0^T e^{(\boldsymbol{Q}-r\boldsymbol{I})t}dt = \left(e^{\boldsymbol{Q}T}e^{-rT} - \boldsymbol{I}\right)\left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}. \tag{4.4}$$

Furthermore,

$$S(T) = \frac{\left(\boldsymbol{\alpha} e^{\boldsymbol{Q}T} e^{-rT} + \boldsymbol{\alpha}\boldsymbol{R}(0,T)r\right)\boldsymbol{\ell}}{\sum_{n=1}^{n_T} e^{-rt_n}\left(1 - \boldsymbol{\alpha} e^{\boldsymbol{Q}t_n}\widehat{\boldsymbol{\ell}}\right)\Delta_n} \tag{4.5}$$

where

$$\widehat{\boldsymbol{\ell}} = \begin{cases} \frac{1}{1-\phi}\boldsymbol{\ell} & \text{if } \phi_1 = \phi_2 = \ldots = \phi_m = \phi \\ \frac{1}{m}\sum_{i=1}^m \boldsymbol{h}^{(i)} & \text{otherwise} \end{cases}. \tag{4.6}$$

Note that the matrix-analytic technique used in Proposition 4.2 has nothing to do with the numerical method chosen to compute the vector $\boldsymbol{\alpha} e^{\boldsymbol{Q}t}$. The matrix-analytic approach uses the analytical features of $e^{\boldsymbol{Q}t}$, in order to simplify probabilistic expression, typically arising in reliability and queuing theory, see e.g. (Neuts 1981), (Neuts 1989), (Assaf, Langbert, Savis & Shaked 1984), (Asmussen 2000) and (Asmussen 2003). For example

$$\int_0^T r_t B_t \mathbb{E}\left[L_t^{(\gamma)}\right]dt = \int_0^T \boldsymbol{\alpha} e^{(\boldsymbol{Q}-r\boldsymbol{I})t}dt\boldsymbol{\ell}^{(\gamma)}r = \left(\boldsymbol{\alpha} e^{\boldsymbol{Q}T}e^{-rT} - \boldsymbol{\alpha}\right)\left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}\boldsymbol{\ell}^{(\gamma)}r$$

which have reduced the computation of the integral to find only $\boldsymbol{\alpha} e^{\boldsymbol{Q}T}$. Another less efficient approach is to consider a discrete approximation of the integral in the left hand side, forcing us to evaluate the vector $\boldsymbol{\alpha} e^{\boldsymbol{Q}t}$ at many time-points t. The matrix-analytic technique will be used several times in this paper, especially in Subsection 5.2 and Subsection 5.3. Other applications of this technique in portfolio credit risk can be found in (Herbertsson 2007) and (Herbertsson 2008).

Recall that $\boldsymbol{\alpha} e^{\boldsymbol{Q}t}$ is the analytical solution of the ODE $\dot{\boldsymbol{p}}(t) = \boldsymbol{p}(t)\boldsymbol{Q}$ with $\boldsymbol{p}(0) = \boldsymbol{\alpha}$ (see (Moeler & Loan 1978)). In our model, this ODE arises due to the Chapmann-Kolmogorov equation, describing the dynamics of the Markov jump process Y_t. Computing $\boldsymbol{\alpha} e^{\boldsymbol{Q}t}$ efficiently is a numerical issue, which for large state spaces requires special treatment, see (Herbertsson & Rootzén 2008). For small state spaces, typically less then 150 states, the task is straightforward using standard mathematical software. There are over 20 different methods of computing the vector $\boldsymbol{\alpha} e^{\boldsymbol{Q}t}$, see (Moeler & Loan 1978) and (Moeler & Loan 2003). One of the these is to solve $\dot{\boldsymbol{p}}(t) = \boldsymbol{p}(t)\boldsymbol{Q}$ by using numerical ODE methods, such as the Runge-Kutta method. This approach is taken by (Frey & Backhaus 2004), (Frey & Backhaus 2008) and (Backhaus 2008), but since they consider time-dependent generators, this will not lead to any simplifications of the spreads as in Proposition 4.2, but only give semi-explicit expressions of these formulas. We will come back to the issue of computing $\boldsymbol{\alpha} e^{\boldsymbol{Q}t}$ in Subsection 6.1, and Subsection 6.2 where we have computed this vector with several different methods.

Other remarks regarding Proposition 4.2 is that finding the generator $\boldsymbol{Q}$ and column vectors $\boldsymbol{\ell}^{(\gamma)}$, $\boldsymbol{\ell}$, $\widehat{\boldsymbol{\ell}}$ are straightforward and the matrix $(\boldsymbol{Q} - r\boldsymbol{I})$ is invertible since it is upper diagonal with strictly negative diagonal elements, see (Herbertsson & Rootzén 2008). Furthermore, several computational shortcuts are possible in Proposition 4.2. The quantities $\boldsymbol{\ell}^{(\gamma)}$, $\boldsymbol{\ell}$ and $\widehat{\boldsymbol{\ell}}$ do not depend on the parametrization, and hence only have to be computed once. The row vectors $\boldsymbol{\alpha} e^{\boldsymbol{Q}T} e^{-rT} + \boldsymbol{\alpha}\boldsymbol{R}(0,T)r$ and $\sum_{n=1}^{n_T} \boldsymbol{\alpha} e^{\boldsymbol{Q}t_n} e^{-rt_n}\Delta_n$ are the same for all CDO tranche spreads and index CDS spreads and hence only have to be computed once for each parametrization determined by (3.1). In particular note that $\sum_{n=1}^{n_T} \boldsymbol{\alpha} e^{\boldsymbol{Q}t_n} e^{-rt_n}\Delta_n$ and $(\boldsymbol{Q} - r\boldsymbol{I})^{-1}$ also appears in the expressions for single-name CDS spreads and k^{th}-to-default spreads studied in (Herbertsson & Rootzén 2008).

In a nonhomogeneous portfolio we have $|\boldsymbol{E}| = 2^m$ which in practice will force us to work with portfolios of size m less or equal to 25, say ((Herbertsson & Rootzén 2008) used $m = 15$). Standard synthetic CDO portfolios typically contains 125 obligors so we will therefore, in Section 5 below, consider a special case of (3.1) which leads to a symmetric portfolio where the state space $\boldsymbol{E}$ can be simplified to make $|\boldsymbol{E}| = m + 1$. This allows us to practically work with the Markov setup in Proposition 4.2 for large m, where $m \geq 125$ with no further complications.

5. A homogeneous portfolio

In this section we apply the results from Section 4 to a homogenous portfolio. First, Subsection 5.1 introduces a symmetric model and shows how it can be applied to price CDO tranche spreads and index CDS spreads. Subsection 5.2 presents formulas for the single-name CDS spread in this model. Finally, Subsection 5.3 is devoted to formulas for k^{th}-to-default swaps on subportfolios of the main portfolio. This problem is slightly different from the corresponding task in previous studies, e.g. (Herbertsson 2005) and (Herbertsson & Rootzén 2008), since the obligors undergo default contagion both from the subportfolio and from obligors outside the subportfolio, in the main portfolio.

5.1. **The homogeneous model for CDO tranches and index CDS-s.** In this subsection we use the results from Section 4 to compute CDO tranche spreads and index CDS spreads in a totally symmetric model. We consider a special case of (3.1) where all obligors have the same default intensities $\lambda_{t,i} = \lambda_t$ specified by parameters a and $b_1, \ldots, b_m$, as

$$\lambda_t = a + \sum_{k=1}^{m-1} b_k 1_{\{T_k \leq t\}} \tag{5.1.1}$$

where $\{T_k\}$ is the ordering of the default times $\{\tau_i\}$ and $\phi_1 = \ldots = \phi_m = \phi$ where ϕ is constant. In this model the obligors are exchangeable. The parameter a is the base intensity for each obligor i, and given that $\tau_i > T_k$, then b_k is how much the default intensity for each remaining obligor jumps at default number k in the portfolio. We start with the simpler version of Proposition 3.1.

Corollary 5.1. *There exists a Markov jump process $(Y_t)_{t\geq 0}$ on a finite state space $\boldsymbol{E} = \{0, 1, 2, \ldots, m\}$, such that the stopping times*

$$T_k = \inf\{t > 0 : Y_t = k\}, \quad k = 1, \ldots, m$$

are the ordering of m exchangeable stopping times $\tau_1, \ldots, \tau_m$ with intensities (5.1.1).

Proof. If $\{T_k\}$ is the ordering of m default times $\{\tau_i\}$ with default intensities $\{\lambda_{t,i}\}$, then the arrival intensity $\lambda_t^{(k)}$ for T_k is zero outside of $\{T_{k-1} \leq t < T_k\}$, otherwise

$$\lambda_t^{(k)} = \left(\sum_{i=1}^{m} \lambda_{t,i}\right) 1_{\{T_{k-1} \leq t < T_k\}}. \tag{5.1.2}$$

Hence, since $\lambda_{t,i} = \lambda_t$ for every obligor i where $\tau_i \geq t$, (5.1.2) implies

$$\lambda_t 1_{\{T_{k-1} \leq t < T_k\}} = \frac{\lambda_t^{(k)}}{m - k + 1}, \quad k = 1, \ldots, m. \tag{5.1.3}$$

Now, let $(Y_t)_{t\geq 0}$ be a Markov jump process on a finite state space $\boldsymbol{E} = \{0, 1, 2, \ldots, m\}$, with generator $\boldsymbol{Q}$ given by

$$\boldsymbol{Q}_{k,k+1} = (m-k)\left(a + \sum_{j=1}^{k} b_j\right) \quad k = 0, 1, \ldots, m-1$$
$$\boldsymbol{Q}_{k,k} = -\boldsymbol{Q}_{k,k+1}, \quad k < m \quad \text{and} \quad \boldsymbol{Q}_{m,m} = 0$$

where the other entries in $\boldsymbol{Q}$ are zero. The Markov process always starts in $\{0\}$ so the initial distribution is $\boldsymbol{\alpha} = (1, 0, \ldots, 0)$. Define the ordered stopping times $\{T_k\}$ as

$$T_k = \inf\{t > 0 : Y_t = k\}, \quad k = 1, \ldots, m.$$

Then, the intensity $\lambda_t^{(k)}$ for T_k on $\{T_{k-1} \leq t < T_k\}$ is given by $\lambda_t^{(k)} = \boldsymbol{Q}_{k-1,k}$. Further, we can without loss of generality assume that $\{T_k\}$ is the ordering of m exchangeable default times $\{\tau_i\}$, with default intensities $\lambda_{t,i} = \lambda_t$ for every obligor i. Hence, if $\tau_i \geq t$, (5.1.3) implies

$$\lambda_t 1_{\{T_{k-1} \leq t < T_k\}} = \frac{\lambda_t^{(k)}}{m-k+1} = \frac{\boldsymbol{Q}_{k-1,k}}{m-k+1} = a + \sum_{j=1}^{k-1} b_j \ , \quad k = 1, \ldots, m$$

and since $\lambda_t = \sum_{k=1}^{m} \lambda_t 1_{\{T_{k-1} \leq t < T_k\}}$, it must hold that $\lambda_t = a + \sum_{k=1}^{m-1} b_k 1_{\{T_k \leq t\}}$, when $\tau_i \geq t$, which proves the corollary. ☐

By Corollary 5.1, the states in $\boldsymbol{E}$ can be interpreted as the number of defaulted obligors in the portfolio.

Recall that the formulas for CDO tranche spreads and index CDS spreads in Proposition 4.2 where derived for an inhomogeneous portfolio with default intensities (3.1). However, it is easy to see that these formulas (with identical recoveries) also can be applied in a homogeneous model specified by (5.1.1), but with $\boldsymbol{\ell}^{(\gamma)}$ and $\boldsymbol{\ell}$ slightly refined to match the homogeneous state space $\boldsymbol{E}$. This refinement is shown in the following lemma.

Lemma 5.2. *Consider a portfolio with m obligors that all satisfy (5.1.1) and let $\boldsymbol{E}$, $\boldsymbol{Q}$ and $\boldsymbol{\alpha}$ be as in Corollary 5.1. Then, (4.2), (4.3) and (4.5) hold, for*

$$\boldsymbol{\ell}_k^{(\gamma)} = \begin{cases} 0 & \text{if} \quad k < n_l(k_{\gamma-1}) \\ k(1-\phi)/m - k_{\gamma-1} & \text{if} \quad n_l(k_{\gamma-1}) \leq k \leq n_u(k_\gamma) \\ \Delta k_\gamma & \text{if} \quad k > n_u(k_\gamma) \end{cases} \tag{5.1.4}$$

where $n_l(x) = \lceil xm/(1-\phi) \rceil$ and $n_u(x) = \lfloor xm/(1-\phi) \rfloor$. Furthermore, $\boldsymbol{\ell}_k = k(1-\phi)/m$.

Proof. Since $L_t = L(Y_t)$ and due to the homogeneous structure, we have

$$\{L_t = k(1-\phi)/m\} = \{Y_t = k\}$$

for each k in $\boldsymbol{E}$. Hence, the loss process L_t is in one-to-one correspondence with the process Y_t. Define $n_l(x) = \lceil xm/(1-\phi) \rceil$ and $n_u(x) = \lfloor xm/(1-\phi) \rfloor$. That is, $n_l(x)$ ($n_u(x)$) is the smallest (biggest) integer bigger (smaller) or equal to $xm/(1-\phi)$. These observations together with the expression for $\boldsymbol{\ell}^{(\gamma)}$ and $\boldsymbol{\ell}$ in Proposition 4.1, yield (5.1.4). □

In the homogeneous model given by (5.1.1), we have now determined all quantities needed to compute CDO tranche spreads and index CDS spreads as specified in Proposition 4.2.

We remark that our symmetric framework is equivalent to the local intensity model which was the starting point in the papers (Schönbucher 2005), (Sidenius, Piterbarg & Andersen 2008), (Lopatin & Misirpashaev 2007) and (Arnsdorf & Halperin 2007). In these articles the authors model the loss-distribution directly by using the so called top-down approach.

5.2. **Pricing single-name CDS in a homogeneous model.** If $F(t)$ is the distribution for τ_i, which by exchangeability is the same for all obligors under (5.1.1), then the single-name CDS spread $R(T)$ is given by (see e.g. (Herbertsson & Rootzén 2008))

$$R(T) = \frac{(1-\phi)\int_0^T B_t dF(t)}{\sum_{n=1}^{n_T} \left(B_{t_n} \Delta_n (1 - F(t_n)) + \int_{t_{n-1}}^{t_n} B_t \left(t - t_{n-1}\right) dF(t) \right)} \tag{5.2.1}$$

where the rest of the notation are the same as in Section 2. Hence, to calibrate, or price single-name CDS-s under (5.1.1), we need the distribution $\mathbb{P}[\tau_i > t]$ (identical for all obligors). This leads to the following lemma.

Lemma 5.3. *Consider m obligors that satisfy (5.1.1). Then, with notation as above*

$$\mathbb{P}[\tau_i > t] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{g} \quad \text{and} \quad \mathbb{P}[T_k > t] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{m}^{(k)}, \quad k = 1, \ldots, m$$

where $\boldsymbol{m}^{(k)}$ and $\boldsymbol{g}$ are column vectors in $\mathbb{R}^{|\boldsymbol{E}|}$ such that $\boldsymbol{m}^{(k)}_j = 1_{\{j<k\}}$ and $\boldsymbol{g}_j = 1 - j/m$.

Proof. By the construction of T_k in Corollary 5.1, we have

$$\mathbb{P}\left[T_k > t\right] = \mathbb{P}\left[Y_t < k\right] = \sum_{j=0}^{k-1} \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{e}_j = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{m}^{(k)} \quad \text{where} \quad \boldsymbol{m}^{(k)}_j = 1_{\{j<k\}}$$

for $k = 1, \ldots, m$. Furthermore, due to the exchangeability,

$$\mathbb{P}\left[T_k > t\right] = \sum_{i=1}^{m} \mathbb{P}\left[T_k > t, T_k = \tau_i\right] = m\mathbb{P}\left[T_k > t, T_k = \tau_i\right]$$

so

$$\mathbb{P}\left[\tau_i > t\right] = \sum_{k=1}^{m} \mathbb{P}\left[T_k > t, T_k = \tau_i\right] = \sum_{k=1}^{m} \frac{1}{m} \mathbb{P}\left[T_k > t\right] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \sum_{k=1}^{m} \frac{1}{m} \boldsymbol{m}^{(k)} = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{g},$$

where $\boldsymbol{g} = \frac{1}{m}\sum_{k=1}^{m} \boldsymbol{m}^{(k)}$. Since $\boldsymbol{m}^{(k)}_j = 1_{\{j<k\}}$ this implies that $\boldsymbol{g}_j = 1 - j/m$ which concludes the proof of the lemma. ☐

A closed-form expression for $R(T)$ is obtained by using Lemma 5.3 in (5.2.1). For ease of reference we exhibit the resulting formulas (proofs can be found in (Herbertsson 2005) or (Herbertsson 2007)).

Proposition 5.4. *Consider m obligors that all satisfies (5.1.1) and assume that the interest rate r is constant. Then, with notation as above*

$$R(T) = \frac{(1-\phi)\boldsymbol{\alpha}\left(\boldsymbol{A}(0) - \boldsymbol{A}(T)\right)\boldsymbol{g}}{\boldsymbol{\alpha}\left(\sum_{n=1}^{n_T}\left(\Delta_n e^{\boldsymbol{Q}t_n} e^{-rt_n} + \boldsymbol{C}(t_{n-1}, t_n)\right)\right)\boldsymbol{g}}$$

where

$$\boldsymbol{C}(s,t) = s\left(\boldsymbol{A}(t) - \boldsymbol{A}(s)\right) - \boldsymbol{B}(t) + \boldsymbol{B}(s), \qquad \boldsymbol{A}(t) = e^{\boldsymbol{Q}t}\left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}\boldsymbol{Q}e^{-rt}$$

and

$$\boldsymbol{B}(t) = e^{\boldsymbol{Q}t}\left(t\boldsymbol{I} + \left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}\right)\left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}\boldsymbol{Q}e^{-rt}.$$

For more on the CDS contract, see e.g (Felsenheimer, Gisdakis & Zaiser 2006), (Herbertsson 2005) or (McNeil et al. 2005).

We remind the reader that in a homogeneous model, the average CDS spread and index CDS spread will coincide if the accrued payment is omitted in the CDS contract. This is not the case in our paper, which implies that we can treat the average CDS and index CDS as two different credit derivatives.

5.3. **Pricing k^{th}-to-default swaps on subportfolios in a homogeneous model.** Consider a homogenous portfolio defined by (5.1.1). Our goal in this subsection is to find expressions for k^{th}-to-default swap spreads on a subportfolio in the main portfolio. The difference in this approach, compared with for example (Herbertsson & Rootzén 2008) and (Frey & Backhaus 2008) is that the obligors undergoes default contagion both from entities in the selected basket and from obligors outside the basket, but in the main portfolio.

Let $\boldsymbol{s}$ be a subportfolio of the main portfolio, that is $\boldsymbol{s} \subseteq \{1, 2, \ldots, m\}$ and let $|\boldsymbol{s}|$ denote the number of obligors in $\boldsymbol{s}$ so $|\boldsymbol{s}| \leq m$. The market standard is $|\boldsymbol{s}| = 5$. If the recoveries are homogeneous, it is enough to find the distribution for the ordering of the default times in the basket. Hence, we seek the distributions of the ordered default times in $\boldsymbol{s}$ denoted by $\{T_k^{(\boldsymbol{s})}\}$. The k^{th}-to-default swap spreads $R_k^{(\boldsymbol{s})}(T)$ on $\boldsymbol{s}$ are then given by (see e.g. (Herbertsson & Rootzén 2008))

$$R_k^{(\boldsymbol{s})}(T) = \frac{(1-\phi)\int_0^T B_t dF_k^{(\boldsymbol{s})}(t)}{\sum_{n=1}^{n_T}\left(B_{t_n}\Delta_n(1-F_k^{(\boldsymbol{s})}(t_n)) + \int_{t_{n-1}}^{t_n} B_t\,(t-t_{n-1})\,dF_k^{(\boldsymbol{s})}(t)\right)} \tag{5.3.1}$$

where $F_k^{(\boldsymbol{s})}(t) = \mathbb{P}\left[T_k^{(\boldsymbol{s})} \leq t\right]$ are the distribution functions for $\{T_k^{\boldsymbol{s}}\}$. The rest of the notation are the same as in Section 2. In Theorem 5.5 below, we derive formulas for the survival distributions of $\{T_k^{(\boldsymbol{s})}\}$. This is done by using the exchangeability, the matrix-analytic approach and the fact that default times in $\boldsymbol{s}$ always coincide with a subsequence of the default times in the main portfolio.

Theorem 5.5. *Consider a portfolio with m obligors that satisfy (5.1.1) and let $\boldsymbol{s}$ be an arbitrary subportfolio with $|\boldsymbol{s}|$ obligors. Then, with notation as above*

$$\mathbb{P}\left[T_k^{(\boldsymbol{s})} > t\right] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t}\boldsymbol{m}^{k,\boldsymbol{s}} \quad \textit{for} \quad k = 1, 2, \ldots, |\boldsymbol{s}| \tag{5.3.2}$$

where

$$\boldsymbol{m}_j^{k,\boldsymbol{s}} = \begin{cases} 1 & \textit{if} \quad j < k \\ 1 - \sum_{\ell=k}^{j\wedge|\boldsymbol{s}|} \frac{\binom{|\boldsymbol{s}|}{\ell}\binom{m-|\boldsymbol{s}|}{j-\ell}}{\binom{m}{j}} & \textit{if} \quad j \geq k. \end{cases} \tag{5.3.3}$$

Proof. The events $\{T_\ell > t\}$ and $\left\{T_k^{(\boldsymbol{s})} = T_\ell\right\}$ are independent where $k \leq \ell \leq m - |\boldsymbol{s}| + k$. To motivate this, note that since all obligors are exchangeable, the information $\left\{T_k^{(\boldsymbol{s})} = T_\ell\right\}$ will not influence the event $\{T_\ell > t\}$. Thus, $\mathbb{P}\left[T_\ell > t, T_k^{(\boldsymbol{s})} = T_\ell\right] = \mathbb{P}\left[T_\ell > t\right]\mathbb{P}\left[T_k^{(\boldsymbol{s})} = T_\ell\right]$. This observations together with Lemma 5.3 implies that

$$\begin{aligned}
\mathbb{P}\left[T_k^{(\boldsymbol{s})} > t\right] &= \sum_{\ell=k}^{m-|\boldsymbol{s}|+k} \mathbb{P}\left[T_k^{(\boldsymbol{s})} > t, T_k^{(\boldsymbol{s})} = T_\ell\right] \\
&= \sum_{\ell=k}^{m-|\boldsymbol{s}|+k} \mathbb{P}\left[T_k^{(\boldsymbol{s})} = T_\ell\right]\mathbb{P}\left[T_\ell > t\right] \\
&= \sum_{\ell=k}^{m-|\boldsymbol{s}|+k} \mathbb{P}\left[T_k^{(\boldsymbol{s})} = T_\ell\right]\boldsymbol{\alpha} e^{\boldsymbol{Q}t}\boldsymbol{m}^{(\ell)} = \boldsymbol{\alpha} e^{\boldsymbol{Q}t}\boldsymbol{m}^{k,\boldsymbol{s}}
\end{aligned}$$

where

$$\boldsymbol{m}^{k,\boldsymbol{s}} = \sum_{\ell=k}^{m-|\boldsymbol{s}|+k} \mathbb{P}\left[T_k^{(\boldsymbol{s})} = T_\ell\right]\boldsymbol{m}^{(\ell)}.$$

Using this and the definition of $\boldsymbol{m}_j^{(\ell)}$ renders

$$\boldsymbol{m}_j^{k,\boldsymbol{s}} = \begin{cases} 1 & \text{if} \quad j < k \\ 1 - \sum_{\ell=k}^{j} \mathbb{P}\left[T_k^{(\boldsymbol{s})} = T_\ell\right] & \text{if} \quad j \geq k \end{cases}$$

and in order to compute $\boldsymbol{m}_j^{k,\boldsymbol{s}}$ for $j \geq k$, note that

$$\bigcup_{\ell=k}^{j} \left\{T_k^{(\boldsymbol{s})} = T_\ell\right\} = \left\{k \leq N_j^{(\boldsymbol{s})} \leq j \wedge |\boldsymbol{s}|\right\}$$

where $N_j^{(\boldsymbol{s})}$ is defined as $N_j^{(\boldsymbol{s})} = \sup\left\{n : T_n^{(\boldsymbol{s})} \leq T_j\right\}$, that is, the number of obligors that have defaulted in the subportfolio $\boldsymbol{s}$ up to the j-th default in the main portfolio. Due to the exchangeability, $N_j^{(s)}$ is a hypergeometric random variable with parameters m, j and $|\boldsymbol{s}|$. Hence,

$$\sum_{\ell=k}^{j} \mathbb{P}\left[T_k^{(\boldsymbol{s})} = T_\ell\right] = \sum_{\ell=k}^{j\wedge|\boldsymbol{s}|} \mathbb{P}\left[N_j^{(\boldsymbol{s})} = \ell\right] = \sum_{\ell=k}^{j\wedge|\boldsymbol{s}|} \frac{\binom{|\boldsymbol{s}|}{\ell}\binom{m-|\boldsymbol{s}|}{j-\ell}}{\binom{m}{j}}.$$

which proves the theorem. □

Returning to k^{th}-to-default swap spreads, expressions for $R_k^{(\boldsymbol{s})}(T)$ may be obtained by inserting (5.3.2) into (5.3.1). The notation and proof are the same as in Proposition 5.4

Corollary 5.6. *Consider a portfolio with m obligors that satisfy (5.1.1) and let $\boldsymbol{s}$ be an arbitrary subportfolio with $|\boldsymbol{s}|$ obligors. Assume that the interest rate r is constant. Then, with notation as above,*

$$R_k^{(\boldsymbol{s})}(T) = \frac{(1-\phi)\boldsymbol{\alpha}\left(\boldsymbol{A}(0) - \boldsymbol{A}(T)\right)\boldsymbol{m}^{k,\boldsymbol{s}}}{\boldsymbol{\alpha}\left(\sum_{n=1}^{n_T}\left(\Delta_n e^{\boldsymbol{Q}t_n} e^{-rt_n} + \boldsymbol{C}(t_{n-1}, t_n)\right)\right)\boldsymbol{m}^{k,\boldsymbol{s}}}, \quad k = 1, 2, \ldots, |\boldsymbol{s}|.$$

For a more detailed description of k^{th}-to-default swap, see e.g. (Felsenheimer et al. 2006), (Herbertsson 2005), (Herbertsson & Rootzén 2008) or (McNeil et al. 2005).

6. Numerical study of a homogeneous portfolio

In this section we calibrate the homogeneous portfolio to real market data on CDO tranches, index CDS-s, average single-name CDS spreads and average FtD-spreads (i.e average 1^{th}-to-default swaps). We match the theoretical spreads against the corresponding market spreads for individual default intensities given by (5.1.1). First, in Subsection 6.1 we give an outline of the calibration technique used in this paper and discuss the two numerical methods used in separate calibrations. Then, in Subsection 6.2 we calibrate our model against an example studied in several articles, e.g (Frey & Backhaus 2008) and (Hull & White 2004), with data from iTraxx Europe, August 4, 2004. The iTraxx Europe spreads has changed drastically in the period between August 2004 and July 2008. We therefore recalibrate our model to a more recent data set, collected at November 28^{th}, 2006 and March 7^{th}, 2008. The last data set was sampled during the subprime-crises. The three calibrations lend some confidence to the robustness of our model. We also give a careful discussion of the outcome of our calibrations with the two different numerical methods used to find the loss-distribution.

Having calibrated the portfolio, we can compute spreads for exotic credit derivatives, not liquidly quoted on the market, as well as other quantities relevant for credit portfolio management. In Subsection 6.3 we compute spreads for tranchelets, which are CDO tranches with smaller loss-intervals than standardized tranches. Subsection 6.4 investigates k^{th}-to-default swap spreads as function of the size of the underlying subportfolio in main calibrated portfolio. Continuing, Subsection 6.5 studies the the implied expected tranche-losses and Subsection 6.6 is devoted to explore the implied loss-distribution.

6.1. **Some remarks on the calibration and numerical implementation.** The symmetric model (5.1.1) can contain at most m different parameters. Our goal is to achieve a "perfect fit" with as many parameters as there are market spreads used in the calibration for a fixed maturity T. For a standard synthetic CDO such as the iTraxx Europe series, we can have 5 tranche spreads, the index CDS spread, the average single-name CDS spread and the average FtD spread. Hence, for calibration, there is at most 8 market prices with maturity T available. However, all of them do not have to be used. We make the following assumption on the parameters b_k for $1 \leq k \leq m-1$

$$b_k = \begin{cases} b^{(1)} & \text{if} \quad 1 \leq k < \mu_1 \\ b^{(2)} & \text{if} \quad \mu_1 \leq k < \mu_2 \\ \vdots & \\ b^{(c)} & \text{if} \quad \mu_{c-1} \leq k < \mu_c = m \end{cases} \tag{6.1.1}$$

where $1, \mu_1, \mu_2, \ldots, \mu_c$ is an partition of $\{1, 2, \ldots, m\}$. This means that all jumps in the intensity at the defaults $1, 2, \ldots, \mu_1 - 1$ are same and given by $b^{(1)}$, all jumps in the intensity at the defaults $\mu_1, \ldots, \mu_2 - 1$ are same and given by $b^{(2)}$ and so on. This is a simple way of reducing the number of unknown parameters from m to $c + 1$.

If η is the number of calibration-instruments, that is the number of credit derivatives used in the calibration, we set $c = \eta - 1$. Let $\boldsymbol{a} = (a, b^{(1)}, \ldots, b^{(c)})$ denote the parameters describing the model and let $\{C_j(T; \boldsymbol{a})\}$ be the η different model spreads for the instruments used in the calibration and $\{C_{j,M}(T)\}$ the corresponding market spreads. In $C_j(T; \boldsymbol{a})$ we have emphasized that the model spreads are functions of $\boldsymbol{a} = (a, b^{(1)}, \ldots, b^{(c)})$ but suppressed the dependence of interest rate, payment frequency, etc. The vector $\boldsymbol{a}$ is then obtained as

$$\boldsymbol{a} = \underset{\hat{\boldsymbol{a}}}{\operatorname{argmin}} \sum_{j=1}^{\eta} \left(C_j(T; \hat{\boldsymbol{a}}) - C_{j,M}(T) \right)^2 \tag{6.1.2}$$

with the constraint that all elements in $\boldsymbol{a}$ are nonnegative. Note that it would have been possible to let the jump parameters b_k be negative, as long as $\lambda_t > 0$ for all t. In economic terms this would mean that the non-defaulted obligors benefit from the default at T_k.

The model-spreads $\{C_j(T; \boldsymbol{a})\}$, such as average CDS spread $R(T; \boldsymbol{a})$, index CDS spread $S(T; \boldsymbol{a})$, CDO tranche spreads $\{S_\gamma(T; \boldsymbol{a})\}$ etc. are given in closed formulas derived in the previous sections. The expressions $\{C_j(T; \boldsymbol{a})\}$ are functionals of the distribution of the

Markov-process Y_t, that is, functions of the vector $\boldsymbol{p}(t) = \boldsymbol{\alpha}e^{\boldsymbol{Q}t}$ at different time points t, where $\boldsymbol{Q}$ in turn can be seen as a function of $\boldsymbol{a}$. Hence, the major challenge lies in computing $\boldsymbol{p}(t)$.

We used two different numerical methods for determining the probability vector $\boldsymbol{p}(t)$. The first method was Padé-approximation with scaling and squaring, (see (Moeler & Loan 1978)) and the second approach was a numerical ODE-solver adapted for stiff ODE system. An ODE-system is called stiff if the absolute values of the eigenvalues of the Jacobian to the system greatly differ in value, that is $\Lambda_{\min} << \Lambda_{\max}$ where $\Lambda_{\min} = \min\{|\Lambda_i|\}$ and $\Lambda_{\max} = \max\{|\Lambda_i|\}$ and $\{\Lambda_i\}$ are the corresponding eigenvalues. In our case the Jacobian is the matrix $\boldsymbol{Q}$ and the eigenvalues $\{\Lambda_i\}$ are given by the diagonal elements in $\boldsymbol{Q}$.

We remind the reader that standard solvers such as the Runge-Kutta method, or any ODE routine not adapted for stiff ODE-solvers are very slow and often inaccurate and can even give raise to instability problems when applied to stiff systems, see e.g. (Enright, Hull & Lindberg 1975) and chapter 9 in (Heath 1996).

We used a numerical-ODE solver adapted to stiff-systems (ode15s in Matlab) which is based on backward differentiation formulas with multistep properties. This solver can be done very fast by exploiting the fact that the Jacobian of our ODE is analytic and simply given by the generator $\boldsymbol{Q}$. Without this observation, the numerical solutions produced by ode15s is much less accurate than the corresponding Padé-solution. The accuracy can be increased by taking smaller time steps and improve the error-tolerance, but with the cost of much longer computational times (of the same order as the running time for the Padé-method). For more on the algorithm used in ode15s, see e.g. (Shampine & Reichelt 1997)

Both the Padé-method and our stiff ODE-solver were applied in separate calibrations in order to verify that the matrix-analytic method is independent of the numerical approach used to compute the model-spreads. This was done for three different data sets. The details of the calibration results are reported in Section 6.2.

The initial parameters in the calibration routine can be rather arbitrary, and the calibrated parameters $\boldsymbol{a}$ are often (but not always) insensitive to variations in the initial parameters. However, this strongly dependent on the number of iterations used in the minimization routine. Furthermore, the more iterations we use, the smaller calibration errors are obtained.

Finally, it should be mentioned that the calibrated parameters $\boldsymbol{a}$ are not likely to be unique. By perturbing the initial guesses, we have been able to get calibrations that are worse, but "close" to the optimal calibration, and where some of the parameters in the calibrated perturbed vector, are very different from the corresponding parameters in the optimal vector. We do not further pursue the discussion of potential nonuniqueness here, but rather conclude that the above phenomena is likely to occur also in other pricing models.

6.2. **Calibration to the iTraxx Europe series and further numerical remarks.** In this subsection we calibrate our model against credit derivatives on the iTraxx Europe series with maturity of five years. There are five different CDO tranche spreads with tranches $[0,3], [3,6], [6,9], [9,12]$ and $[12,22]$, and we also have the index CDS spreads and the average CDS spread.

Table 1. iTraxx Europe Series 3, 6 and 8 collected at August 4^{th} 2004, November 28^{th}, 2006 and March 7^{th}, 2008. The market and model spreads and the corresponding absolute errors, both in bp and in percent of the market spread. The $[0,3]$ spread is quoted in %. All maturities are for five years.

2004-08-04	Market	Model	error (bp)	error (%)
$[0,3]$	27.6	27.6	3.851e-005	1.4e-006
$[3,6]$	168	168	0.000316	0.0001881
$[6,9]$	70	70	0.000498	0.0007115
$[9,12]$	43	43	0.0005563	0.001294
$[12,22]$	20	20	0.0004006	0.002003
index	42	42.02	0.01853	0.04413
avg CDS	42	41.98	0.01884	0.04486
Σ abs.cal.err		0.03918 bp		

2006-11-28	Market	Model	error (bp)	error (%)
$[0,3]$	14.5	14.5	0.008273	0.0005705
$[3,6]$	62.5	62.48	0.02224	0.03558
$[6,9]$	18	18.07	0.07275	0.4042
$[9,12]$	7	6.872	0.1282	1.831
$[12,22]$	3	3.417	0.4169	13.9
index	26	26.15	0.1464	0.5632
avg CDS	26.87	26.13	0.7396	2.752
Σ abs.cal.err		1.534 bp		

2008-03-07	Market	Model	error (bp)	error (%)
$[0,3]$	46.5	46.5	0.0505	0.001086
$[3,6]$	567.5	568	0.4742	0.08356
$[6,9]$	370	370	0.04852	0.01311
$[9,12]$	235	234	1.035	0.4404
$[12,22]$	145	149.9	4.911	3.387
index	150.3	144.3	5.977	3.978
avg CDS	145.1	143.8	1.296	0.8933
Σ abs.cal.err		13.79 bp		

First, a calibration is done against data taken from iTraxx Europe on August 4, 2004 used in e.g. (Frey & Backhaus 2008) and (Hull & White 2004). Here, just as in (Frey & Backhaus 2008) and (Hull & White 2004), we set the average CDS spread equal to (i.e. approximated by) the index CDS spread. No market data on FtD spreads are available in this case. The iTraxx Europe spreads has changed drastically since August 2004. We therefore recalibrate our model to some more recent data sets, collected at November 28^{th}, 2006 and March 7^{th}, 2008. These data sets also contains the average CDS spread. The November 28^{th}, 2006 sample also contains the average FtD spread (see Table 9). All data is taken from `Reuters` on November 28^{th}, 2006 and March 7^{th}, 2008 and the bid, ask and mid spreads are displayed in Table 7 and Table 8.

In all three calibrations the interest rate was set to 3%, the payment frequency was quarterly and the recovery rate was 40%. We choose the partition $\mu_1, \mu_2, \ldots, \mu_6$ so that it roughly coincides with the number of defaults needed to reach the upper attachment point for each tranche, see Table 10 in Appendix.

Since the ODE method is around 10 times faster than the Padé approach, we used the stiff-ODE solver to find the optimal parameter in the calibration routine. This algorithm can therefore perform 10 times more iterations than the Padé-approach, in the same time-span. We used 1000 iterations in our optimization (i.e calibration) routine, and the numerical values of the calibrated parameters $\boldsymbol{a}$, obtained via (6.1.2), are shown in Table 11 in Appendix 8.

In all three data-sets we obtained perfect fits, although in the 2008 portfolio the accumulated calibration error was around 9 times higher compared to the 2006 portfolio. The relative calibration errors were however very good. Furthermore, some of the corresponding spreads in the 2008 data-set had increased a factor 50 compared to 2006 portfolio, see Table 1.

Once we had found the optimal calibrated parameters with the ODE-solver, we used these parameter to compute the model-spreads also with the Padé-method. This in order to compare the two numerical approaches for given parameters $\boldsymbol{a}$. We did this with all data sets and found that the relative error between the ODE and Padé-model spreads (in terms of the ODE-case) never exceed 0.0001%. Hence, for a given set of parameters $\boldsymbol{a}$, the two different numerical methods seem to produce almost identical results.

6.2.1. *Explosion of the last jump-parameter.* We note that in the 2008-03-07 portfolio, the jump-parameter $b^{(6)}$ "explodes" (see Table 11 in Appendix 8) compared to the other parameters in the vector $\boldsymbol{a}$. A similar behavior is also seen 2006-11-28 portfolio, but more moderate than in the 2008-03-07 case. By construction, $b^{(6)}$ only affects tranches above 22%. Hence, in our case only the index-CDS spread and the average-CDS spread are affected by $b^{(6)}$. In order to investigate how the jump-parameter $b^{(6)}$ influence these two spreads in the 2008-03-07 portfolio, we changed $b^{(6)}$ from 78 (see Table 11) to 6.5, 2.5 and 1.5, holding the other parameters in $\boldsymbol{a}$ fixed, and then computed the new spreads. The changes in the spreads were negligible, as reported in Table 12. In Section 6.6 we continue to study the impact of the explosion in the jump-parameter $b^{(6)}$ on our model-quantities.

6.2.2. *Monte Carlo simulations.* With the parameters in Table 11, we also determined the credit spreads by using Monte Carlo simulations and compared these with the model-spreads computed with the ODE-routine. The relative differences in all three data-sets did not exceed 1.38 %, where we used 10^6 replications in the simulation, see Table 14 in Appendix. This test also lends some confidence in the correctness of the implementations of the two different deterministic numerical methods used to find distribution of the Markov process.

6.2.3. *Comparing calibrations:Padé-method vs. ODE-solver.* In the calibration yielding Table 1 we used the stiff ODE-solver with 1000 iterations in the minimization routine, to find the optimal parameters $\boldsymbol{a}$. However, it is of great interest to also conduct the calibration with the Padé-approach and then compare the results with those obtained with the ODE-routine. For our three data sets, we therefore performed calibrations both with the Padé-approach and the ODE-method and compared the errors as well as the calibrated parameters $\boldsymbol{a}$. Furthermore, for each numerical approach and each data set, we used the same initial parameters in the calibration routine and the same number of iterations (100 iterations).

In the 2008 portfolio the accumulated calibration errors where 18.4 bp for the ODE-approach and and 21.9 bp for the Padé approximation. The calibrated parameters in the vector $\boldsymbol{a}$ obtained with the two methods did not differ more than 5.8 %, except for $b^{(3)}$ and $b^{(6)}$ which still was in the same order, see Table 13. The parameter $b^{(4)}$ was of the order 10^{-12} for the ODE-case and 10^{-7} in the Padé-method, i.e. close to zero in both methods, and is therefore not relevant when compared with the other parameters in $\boldsymbol{a}$.

In the 2004 and 2006 data-sets we observed a similar behavior between the calibrated parameters retrieved from the two approaches, and the accumulated calibration errors where almost identical for both numerical-approaches.

6.2.4. *Padé-method or ODE-solver? The matrix-analytic method is independent of the numerical approach.* All of the above studies gives evidence to the obvious fact that using the matrix-analytic approach to find the spreads for credit derivatives, is independent of the numerical approach chosen to compute the probability vector $\boldsymbol{p}(t) = \boldsymbol{\alpha} e^{\boldsymbol{Q}t}$. The small numerical differences in the model-spreads which arise using different methods to find $\boldsymbol{p}(t)$ have to be attributed to the intrinsic differences in the corresponding algorithms used, i.e. the stiff ODE-solver and the Padé-approximation method.

It is difficult to determine which of the two methods that is optimal from an overall point of view. First, we remind the reader that standard solvers such as the Runge-Kutta method, or any ODE routine not adapted for stiff ODE-solvers are outperformed by the Padé approximation on all levels, such as computational time, accuracy of the solution, analytical error-control etc. Secondly, even though numerical ODE-solvers for stiff problems and the Padé-approach are implemented in some mathematical-software packages, it is still important to compare the required amount of work needed to implement each of these two methods. Conditional on the fact that both methods need matrix-packages, the Padé-approach can be implemented in very few rows using built in matrix-multiplications (19 rows in matlab). On the other hand, implementing a stiff ODE-solver that uses backward differentiation formulas with multistep properties together with analytical Jacobian techniques requires a huge among of coding compared to the Padé-approach. The main reason is that a general numerical ODE-solver do not exploit the analytical features of an ODE-system with constant parameters which leads to the analytical solution $\boldsymbol{p}(t) = \boldsymbol{\alpha} e^{\boldsymbol{Q}t}$. This remark has also been done on p.122 in (Moeler & Loan 1978).

6.3. **Pricing tranchelets in a homogeneous model.** As discussed above, a tranchelet is a nonstandard CDO tranche with smaller loss-intervals than standardized tranches, see e.g. (Brigo, Pallavicini & Torresetti 2006) or (JPMorgan 2006). Tranchelets are typically computed for losses on $[0, 1], [1, 2], \ldots, [5, 6]$. Currently, there are no liquid market for these instruments, so they can still be regarded as somewhat "exotic". Nevertheless, tranchelets have recently become popular and pricing these instruments are done in the same ways as for standard tranches.

Table 2. Tranchelet spreads on iTraxx Europe, November 28^{th} 2006 (Series 6) and March 7^{th}, 2008 (series 8) and the absolute difference in % of the 2006-11-28 spreads. The $[0, 1]$ and $[1, 2]$ spreads are the upfront premiums on the tranche nominals, quoted in % where the running fee is 500 bp. Tranchelets above $[1, 2]$ are expressed in bp. All maturities are five years.

Tranchelet	2006/11/28	2008/03/07	diff. (in %)
$[0, 1]$	47.89	73.39	53.24
$[1, 2]$	7.016	44.28	531.1
$[2, 3]$	245.8	1050	327.1
$[3, 4]$	98.02	682.4	596.2
$[4, 5]$	54.53	549.7	908.0
$[5, 6]$	35.12	475.4	1254
$[6, 7]$	24.23	420.5	1635
$[7, 8]$	17.32	369.3	2032
$[8, 9]$	12.68	320.8	2431
$[9, 10]$	9.305	275.2	2858
$[10, 11]$	6.668	232.9	3393
$[11, 12]$	4.644	194.1	4079

In this subsection we compute the five year tranchelet spreads for $[0, 1], \ldots, [11, 12]$, on iTraxx Europe Series 6, November 28^{th} 2006, and iTraxx Europe Series 8, March 7^{th}, 2008 as well as the corresponding absolute difference in % of the 2006-11-28 spreads. The computations are done with parameters obtained from the calibrations in Table 1 where all other quantities such as recovery rate, interest rate, payment frequency etc. are the same as in these tables. The $[0, 1]$ and $[1, 2]$ spreads are computed with Equation (4.3) where $\boldsymbol{\ell}^{(1)}$ is replaced by a corresponding column vector adapted for $[0, 1]$, and $[1, 2]$ respectively, given as in Lemma 5.2. Furthermore, in (4.3), k_1 is set to 0.01 for both tranchelets $[0, 1]$ and $[1, 2]$. Tranchelets above $[1, 2]$ are computed with Equation (4.2).

It is interesting to note that the average for the three tranchelets between 3 and 6 are 62.56 (2006-11-28) and 569.2 (2008-03-07) which both are close to the corresponding $[3, 6]$ spreads. The same holds for the averages of tranchlets between 6 to 9 and 9 to 12, which are 18.08, 370.2 and 6.872, 234.1 respectively. These observations explain why the average of the differences for the three tranchelets between 3 to 6, 6 to 9 and 9 to 12, given by

Table 3. The market spreads (used for calibration) on iTraxx Europe, November 28^{th} 2006 (Series 6) and March 7^{th}, 2008 (series 8) and the absolute difference in % of the 2006-11-28 spreads. All maturities are five years.

	$[0,3]$	$[3,6]$	$[6,9]$	$[9,12]$	$[12,22]$	index	avg CDS
2006/11/28	14.5	62.5	18	7	3	26	26.87
2008/03/07	46.5	567.5	370	235	145	150.3	145.1
diff. (%)	220.7	808	1956	3257	4733	477.9	440.1

919%, 2033 % and 3443 %, are close to the corresponding differences in the $[3,6]$, $[6,9]$ and $[9,12]$ tranche spreads, displayed in Table 3.

6.4. **Pricing k^{th}-to-default swaps on subportfolios in a homogeneous model.** In this subsection we price five year k^{th}-to-default spreads $R_k^{(\boldsymbol{s})}$ with $k = 1, \ldots, 5$ for different subportfolios $\boldsymbol{s}$, of the main portfolio. The subportfolios have sizes $|\boldsymbol{s}| = 5, 10, 15, 25, 30$ and the computations are done for the two different data sets, iTraxx Europe Series 6, November 28^{th}, 2006 and iTraxx Europe Series 8, March 7^{th}, 2008. The computations are done with parameters obtained from the calibrations in Table 1, where all other quantities such as recovery rate, interest rate, payment frequency etc. are the same as in these tables.

Table 4. The five year k^{th}-to-default spreads $R_k^{(\boldsymbol{s})}$ with $k = 1, \ldots, 5$ for different subportfolios $\boldsymbol{s}$ in the main portfolio calibrated to iTraxx Europe, November 28^{th} 2006 (Series 6) and March 7^{th}, 2008 (series 8) and the absolute difference in % of the 2006-11-28 spreads. We consider $|\boldsymbol{s}| = 5, 10, 15, 25, 30$.

$\|\boldsymbol{s}\|$	Date	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
5	2006/11/28	119	9.61	2.335	1.764	1.639
	2008/03/07	357.4	127.3	88.66	82.34	81.61
	diff. (%)	200.4	1224	3698	4567	4879
10	2006/11/28	226.7	30.6	6.191	2.616	1.96
	2008/03/07	573.9	217.7	125.5	94.95	85.11
	diff. (%)	153.2	611.5	1927	3530	4243
15	2006/11/28	327.4	58.88	13.69	4.853	2.692
	2008/03/07	759	310.4	178	123.5	98.56
	diff. (%)	131.8	427.3	1200	2444	3561
20	2006/11/28	422.8	91.71	24.34	8.69	4.238
	2008/03/07	925.7	398.7	233.6	160.9	121.9
	diff. (%)	118.9	334.7	859.9	1751	2775
25	2006/11/28	513.7	127.6	37.61	14	6.69
	2008/03/07	1080	482.8	287.7	200.7	151.3
	diff. (%)	110.3	278.4	665	1334	2162

There exists liquid quoted market spreads on FtD baskets (i.e. $k = 1$) and often the FtD spreads are also quoted in percent of the sum of the individual spreads in the subportfolio $\boldsymbol{s}$ (see Table 9 in Appendix). For the 2006-11-28 case the model FtD-spread is 119.9 bp which is close to the observed average mid FtD-spread given by 116.8 bp, see in Table 9.

From Table 4 we see that, for fixed $\boldsymbol{s}$ and k, the spreads differ substantially between the two dates. Given the difference between the market spreads in the calibration (Table 3), this should not come as a surprise. For example, when $|\boldsymbol{s}| = 5$, $k = 1$ the difference is 200 %, and for $|\boldsymbol{s}| = 15$, $k = 5$ the 2008-03-07 spread is around 3560 % bigger than the 2006-11-28 spread. The spreads increase as the size of the portfolio increases, as they should.

For the 2006-11-28 case, the increase from a portfolio of size 5 to one of size 25 is 432% for a 1^{st}-to-default swap, 1330% for a 2^{nd}-to-default swap, 1628% for a 3^{rd}-to-default swap, and for a 5^{th}-to-default swap the increase is 421%. Further, for a portfolio of size 10 the price of a 1^{st}-to-default swap is about 117 times higher than for a 5^{th}-to-default swap and the corresponding ratio for a portfolio of size 15 is about 122. These ratios are much smaller than for a "isolated" portfolio, which only undergo default contagion from obligors within the basket, see (Herbertsson & Rootzén 2008). Qualitatively the above results are completely as expected, however, given market spreads on CDO tranches, index CDS spreads etc. it would seem rather impossible to guess the sizes of the effects without computation.

6.5. **The implied tranche losses in a homogeneous portfolio.** In the credit literature today, expected risk-neutral tranche losses are often called *implied* tranche losses. Here "implied" is refereing to the fact that the quantities are retrieved from market data via a model. Similarly, the implied portfolio loss refers to the expected risk-neutral portfolio loss. In this subsection we compute the implied expected tranche losses. These are important

Table 5. The implied tranche losses in % of tranche nominal, at $t = 5$ for the calibrated CDO portfolios on iTraxx Europe, November 28^{th} 2006 (Series 6) and March 7^{th}, 2008 (series 8) and the absolute difference in % of the 2006-11-28 tranche losses.

	$[0,3]$	$[3,6]$	$[6,9]$	$[9,12]$	$[12,22]$
2006/11/28	36.59	3.257	0.9526	0.3636	0.1812
2008/03/07	67.15	27.66	18.66	12.05	7.815
diff. (%)	83.49	749.3	1859	3214	4213

quantities for a credit manager and Lemma 4.1 and Lemma 5.2 provides formulas for computing them. We study $100 \cdot \mathbb{E}\left[L_5^{(\gamma)}\right] / \Delta k_\gamma$ on CDO portfolios calibrated against iTraxx Europe Series 6, November 28^{th} 2006, and iTraxx Europe Series 8, March 7^{th}, 2008. Just as for previous computations, the corresponding tranche losses differ substantially between the two dates. For example, in the 2008-03-07 case, the tranche loss on $[0, 3]$ has increased 83 % relatively the 2006-11-28 portfolio. But this differences drastically increases for the upper tranches, $[6,9],[9,12]$ to 1859% and 3214%, as seen in Table 5. Hence, on March 7 2008, the five year implied expected tranche losses on $[6,9]$ and $[9,12]$ had increased a factor 19 and 33 respectively, compared to the corresponding values for the November 28^{th} 2006 portfolio.

6.6. **The implied loss distribution in a homogeneous portfolio.** In this subsection we study the implied distribution for the loss process L_t. Since we are considering constant recovery rates, then for every t, the distribution of L_t is discrete and formally the values for $\mathbb{P}[L_t = x]$ should be displayed as bars at $x = k(1-\phi)/m$ where $0 \leq k \leq m$. However, since there are totaly 126 different outcomes we do not bother about this and connect the graph continuously between each discrete probability. The loss probabilities are computed by using that $L_t = L(Y_t)$ so $\mathbb{P}[L_t = k(1-\phi)/m] = \mathbb{P}[Y_t = k] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t}\boldsymbol{e}_k$ for $k = 0, 1, \ldots, m$, see Corollary 5.1.

We study the implied loss distribution on the interval $0 \leq x \leq 22\%$ for $t = 5$, see Figure 1. In order to also get a grasp of the implied distribution for bigger losses, we have displayed $\mathbb{P}[L_5 \geq x\%]$ for $x = 3, 6, 9, 12, 22$ and $x = 60$, in Table 6. Note that with 40% recovery, $\mathbb{P}[L_5 \geq 60\%] = \mathbb{P}[L_5 = 60\%] = \mathbb{P}[Y_5 = 125]$ is the so called five-year "Armageddon probabilities", i.e. the probability that all obligors in the portfolio have defaulted within 5 years from the date the portfolio was calibrated. The five year "Armageddon probabilities" are negligible for the 2004 and 2006 portfolios (0.08 % and 0.127 respectively). However, this is not the case for the 2008 data-set, where $\mathbb{P}[L_5 = 60\%] = 7.11\%$, that is, there is 7% probability that all 125 obligors in the portfolio have defaulted within 5 years from March 2008. In reality, this will likely not happen, since risk-neutral (implied) default probabilities are substantially larger than the "real", so called actuarial, default probabilities. The big differences in $\mathbb{P}[L_5 = 60\%]$, between the 2006 and 2008 portfolios are most likely due to the subprime-crises, that emerged 2007 and continued into 2008.

For our three portfolios, we note that $\mathbb{P}[22\% < L_5 < 60\%] = \mathbb{P}[L_5 \geq 22\%] - \mathbb{P}[L_5 \geq 60\%]$ is equal to 0.3685%, 0.0409% and 0.0137% respectively, which are negligible compared to the probabilities $\mathbb{P}[x_1\% < L_5 < x_2\%]$ for $[x_1, x_2] = [3,6], [6,9], [9,12]$ and $[12,22]$ in each portfolio. Hence, the market implies a negligible probability for having a loss bigger than 22 % but strictly smaller than 60 %, within five years from the corresponding sample-date.

Table 6. The probabilities $\mathbb{P}[L_5 \geq x\%]$ (in %) where $x = 3, 6, 9, 12, 22$ and $x = 60$, for the 2004-08-04, 2006-11-28 and 2008-03-07 portfolios.

$\mathbb{P}[L_5 \geq x\%]$	$x=3$	$x=6$	$x=9$	$x=12$	$x=22$	$x=60$
2004/08/04	14.7	4.976	2.793	1.938	0.4485	0.07997
2006/11/28	6.466	1.509	0.5935	0.2212	0.1674	0.1265
2008/03/07	35.67	22.26	15.44	9.552	7.122	7.108

Finally, we also studied how the jump-parameter $b^{(6)}$ influence the probabilities $\mathbb{P}[L_5 \geq x\%]$. To be more specific, in the 2008-03-07 portfolio, we changed $b^{(6)}$ from 78 (see Table 11) to 6.5, 2.5 and 1.5, holding the other parameters in $\boldsymbol{a}$ fixed, and then computed $\mathbb{P}[L_5 \geq x\%]$

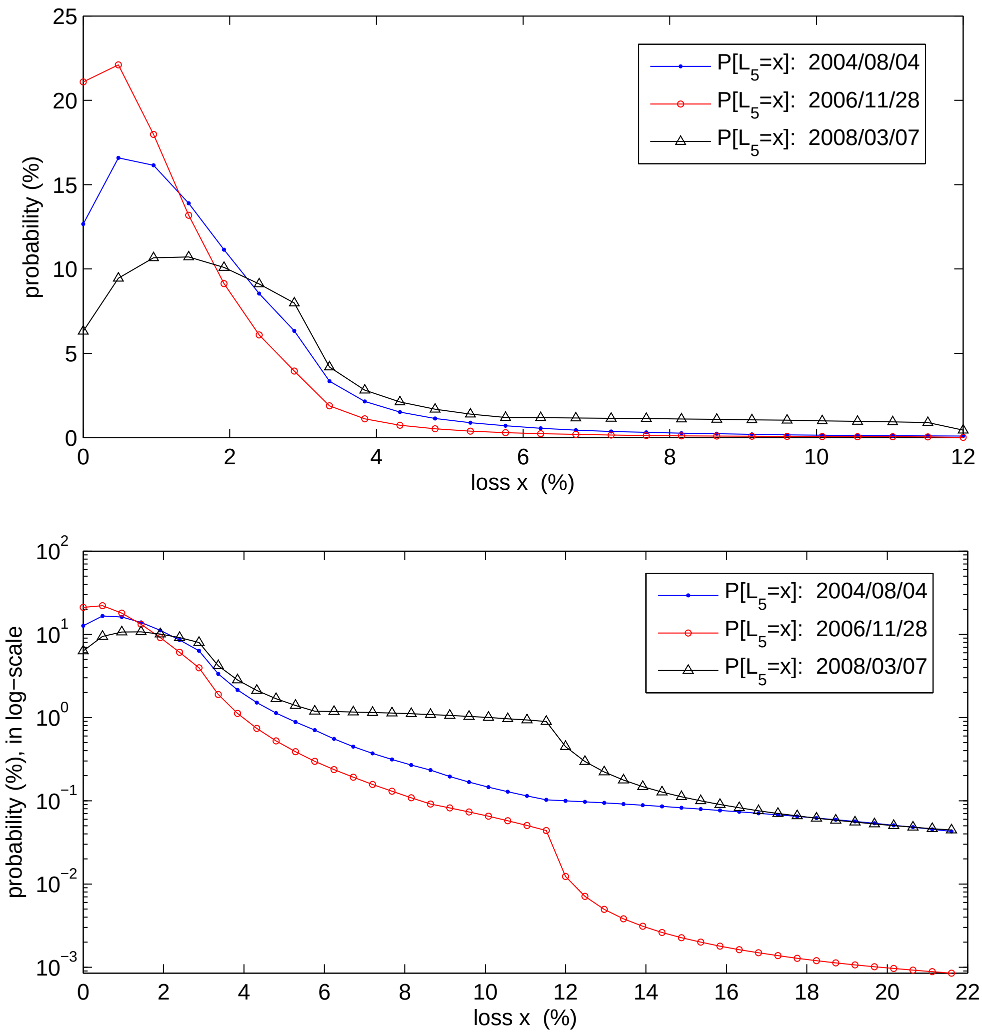

Figure 1. The five year implied loss distributions $\mathbb{P}[L_5 = x\%]$ (in %) for the 2004-08-04, 2006-11-28 and 2008-03-07 portfolios, where $0 \leq x \leq 12$ (upper) and $0 \leq x < 22$ (lower). The lowers graph is in log-scale.

for the same x-s as in Table 6. The probabilities $\mathbb{P}[L_5 \geq x\%]$ did not change except for $x = 60$ which now rendered $\mathbb{P}[L_5 = 60\%] = 6.966\%, 6.745\%$ and 6.528%. This represents a relative difference of 2.00%, 5.11% and 8.15% in terms of case when $b^{(6)} = 78$, i.e. when $\mathbb{P}[L_5 = 60\%] = 7.108\%$. The above results imply that $\mathbb{P}[22\% < L_5 < 60\%]$ is still negligible compared with $\mathbb{P}[x_1\% < L_5 < x_2\%]$ for $[x_1, x_2] = [3, 6], [6, 9], [9, 12]$ and $[12, 22]$. Our findings are consistent with the discussion from Subsection 6.2.1, where we concluded that the model is rather insensitive to explosions in the jump-parameter $b^{(6)}$.

7. Conclusions

In this paper we have derived closed-form expressions for CDO tranche spreads and index CDS spreads. This is done in an inhomogeneous model where dynamic default dependencies among obligors are expressed in an intuitive, direct and compact way. By specializing this model to a homogenous portfolio, we show that the CDO and index CDS formulas simplify considerably in a symmetric model. The same method are used to derive k^{th}-to-default swap spreads for subportfolios in the main CDO portfolio. In this setting, we calibrated a symmetric portfolio against credit derivatives on the iTraxx Europe series for a fixed maturity of five years. We did this at three different dates, where the corresponding market spreads differ substantially. In all three cases we obtained perfect fits. We also implemented two different numerical methods, an ODE-solver and Padé-approximation, to determine the distribution of the Markov-process. Both of the methods were applied in separate calibrations in order to verify that the matrix-analytic method is independent of the numerical approach used to find the law of the Markov process. Consequently, applying the matrix-analytic method to find the credit derivative model-spreads are therefore independent of the numerical approach used to compute the loss-probabilities. The computations were complemented with Monte Carlo simulations, for all three data-sets, in order to check the correctness of the numerical implementations. These studies therefore lend some confidence to the robustness of our model.

In the calibrated portfolios, we computed tranchelet spreads and investigated k^{th}-to-default swap spreads as function of the portfolio size. Further, the implied tranche losses and the implied loss distributions were also extracted. All these computations and investigations would be difficult to perform without having convenient formulas for the quantities that we want to study. Furthermore, given the recovery rate, the number of model parameters are as many as the market instruments used in the calibration. This implies that all calibrations are performed without inserting "fictitious" numerical values for some of the parameters, making the calibration more realistic.

Finally, we remark that our symmetric framework is equivalent to the local intensity model.

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8. Appendix

Below we state the proof of Proposition 4.2.

Proof. Since $r_t = r$, using Lemma 4.1 we have that

$$\int_0^T r_t B_t \mathbb{E}\left[L_t^{(\gamma)}\right] dt = \boldsymbol{\alpha} \int_0^T e^{(\boldsymbol{Q}-r\boldsymbol{I})t} dt \boldsymbol{\ell}^{(\gamma)} r = \boldsymbol{\alpha}\boldsymbol{R}(0,T)\boldsymbol{\ell}^{(\gamma)} r$$

where $\boldsymbol{R}(0,T)$ is given by (4.4). So by Lemma 4.1 again, we get

$$V_\gamma(T) = B_T \mathbb{E}\left[L_T^{(\gamma)}\right] + \int_0^T r_t B_t \mathbb{E}\left[L_t^{(\gamma)}\right] dt = \left(\boldsymbol{\alpha} e^{\boldsymbol{Q}T} e^{-rT} + \boldsymbol{\alpha}\boldsymbol{R}(0,T) r\right) \boldsymbol{\ell}^{(\gamma)}$$

and

$$W_\gamma(T) = S_\gamma(T) \sum_{n=1}^{n_T} B_{t_n} \left(\Delta k_\gamma - \mathbb{E}\left[L_{t_n}^{(\gamma)}\right]\right) \Delta_n = S_\gamma(T) \sum_{n=1}^{n_T} e^{-rt_n} \left(\Delta k_\gamma - \boldsymbol{\alpha} e^{\boldsymbol{Q}t_n} \boldsymbol{\ell}^{(\gamma)}\right) \Delta_n.$$

Recall that for all tranches γ, except for the equity tranche, the spreads $S_\gamma(T)$ are determined so that $V_\gamma(T) = W_\gamma(T)$. Thus, the equations above prove (4.2). Furthermore, for the equity tranche, $S_1(T)$ is set to 500 bp and the up-front premium $S_1^{(u)}(T)$ is determined so that $V_1(T) = S_1^{(u)}(T) k_1 + W_1(T)$. The expressions for $V_1(T)$ and $W_1(T)$ together with the fact that $\Delta k_1 = k_1$ then imply that $S_1^{(u)}(T)$ is given by

$$\begin{aligned} S_1^{(u)}(T) &= \frac{1}{k_1}\left[B_T\mathbb{E}\left[L_T^{(1)}\right] + \int_0^T r_t B_t \mathbb{E}\left[L_t^{(1)}\right]dt - 0.05\sum_{n=1}^{n_T} B_{t_n}\left(\Delta k_1 - \mathbb{E}\left[L_{t_n}^{(1)}\right]\right)\Delta_n\right] \\ &= \frac{1}{k_1}\left[\left(\boldsymbol{\alpha} e^{\boldsymbol{Q}T}e^{-rT} + \boldsymbol{\alpha}\boldsymbol{R}(0,T)r\right)\boldsymbol{\ell}^{(1)} - 0.05\sum_{n=1}^{n_T} e^{-rt_n}\left(\Delta k_1 - \boldsymbol{\alpha} e^{\boldsymbol{Q}t_n}\boldsymbol{\ell}^{(1)}\right)\Delta_n\right] \\ &= \frac{1}{k_1}\left(\boldsymbol{\alpha} e^{\boldsymbol{Q}T}e^{-rT} + \boldsymbol{\alpha}\boldsymbol{R}(0,T)r + 0.05\sum_{n=1}^{n_T}\boldsymbol{\alpha} e^{\boldsymbol{Q}t_n}e^{-rt_n}\Delta_n\right)\boldsymbol{\ell}^{(1)} - 0.05\sum_{n=1}^{n_T} e^{-rt_n}\Delta_n \end{aligned}$$

which establish Equation (4.3). Finally, to find expressions for the index CDS spreads $S(T)$, recall that this contract is almost identical to a CDO tranche (see (2.3.1)), with the differences that $\boldsymbol{\ell}^{(\gamma)}$ is replaced by $\boldsymbol{\ell}$ in the protection leg, and in the premium leg Δk_γ is replaced by 1 and $\boldsymbol{\ell}^{(\gamma)}$ by $\widehat{\boldsymbol{\ell}}$, where

$$\widehat{\boldsymbol{\ell}} = \begin{cases} \frac{1}{1-\phi}\boldsymbol{\ell} & \text{if } \phi_1 = \phi_2 = \ldots = \phi_m = \phi \\ \frac{1}{m}\sum_{i=1}^m \boldsymbol{h}^{(i)} & \text{otherwise} \end{cases}$$

which proves Equation (4.5) and (4.6). □

Tables 7 and Table 8 show the market spreads collected from iTraxx Europe Series 6, November 28^{th}, 2006 and iTraxx Europe Series 8, March 7^{th}, 2008. Both data-sets are sampled from `Reuters`.

Table 9 shows the FtD spreads, i.e. 1^{st}-to-defaults spreads for 6 standardized subportfolios on iTraxx Europe Series 6, launched September 20^{th}, 2006. Each basket consist of five obligors that are taken from a sector in the iTraxx Series 6 (Autos, Energy, Industrial, TMT, Consumers and Financial). The names of the obligors in each basket as well as the selection criteria can be found on the webpage for iboxx. In the financial FtD basket, we have used the subordinated FtD spread, since the senior spread is much smaller (30 bp) than the other spreads, which will pull down the average mid FtD spread to 112.25 bp.

The numerical values of the calibrated parameters $\boldsymbol{a}$, obtained via (6.1.2), are shown in Table 11 and the partition (see Equation (6.1.1)) in Table 10.

Table 13 gives the calibrated parameters $\boldsymbol{a}$ as a function of the the numerical method used to compute the distribution for the Markov process, in the calibration routine. In both cases the same initial parameters are used in the calibration, and the same number of iterations (100 iterations).

In Table 12 we display the index-CDS spread and average CDS-spread in the 2008-03-07 portfolio, as function of the jump parameter $b^{(6)}$, when holding the other parameters in $\boldsymbol{a}$ fixed.

Finally, Table 14 gives the Monte Carlo spreads compared to the model-spreads, in our three data-sets.

Table 7. The market bid, ask and mid spreads for iTraxx Europe (Series 6), November 28^{th}, 2006. All data is taken from Reuters. The mid spreads, i.e. average of the bid and ask spread, are used in the calibration in Section 6.

	bid	ask	mid	time
$[0, 3]$		14.5	14.5	28 Nov, 18:23
$[3, 6]$	60	65	62.5	28 Nov, 17:14
$[6, 9]$	16.5	19.5	18	28 Nov, 13:36
$[9, 12]$	5.5	8.5	7	28 Nov, 13:36
$[12, 22]$	2	4	3	28 Nov, 13:36
index	25.75	26.25	26	28 Nov, 18:34
avg CDS	25.94	27.8	26.87	28 Nov, 19:40

Table 8. The market bid, ask and mid spreads for iTraxx Europe (Series 8), March 7^{th}, 2008. All data is taken from Reuters. The mid spreads, i.e. average of the bid and ask spread, are used in the calibration in Section 6.

	bid	ask	mid	time
$[0, 3]$	46.5	46.5	46.5	March 7, 16:58
$[3, 6]$	550	585	567.5	March 7, 16:53
$[6, 9]$	350	390	370	March 7, 10:32
$[9, 12]$	225	245	235	March 7, 14:54
$[12, 22]$	125	165	145	March 7, 12:07
index	150	150.5	150.3	March 7, 18:39
avg CDS	141.1	149.1	145.1	March 7,

Table 9. The market bid, ask and mid spreads for different FtD spreads on subsectors of iTraxx Europe (Series 6), November 28^{th}, 2006. Each subportfolio have five obligors. We also display the sum of CDS-spreads (SoS) in each basket, as well as the mid FtD spreads in % of SoS.

Sector	bid	ask	mid	SoS	mid/SoS %	time
Autos	154	166	160	202	79.21 %	28 Nov, 10:26
Energy	65	71	68	86	79.07 %	28 Nov, 10:26
Industrial	114	123	118.5	141	84.04 %	28 Nov, 10:26
TMT	167	188	177.5	217	81.8 %	28 Nov, 10:26
Consumers	113	122	117.5	140	83.93 %	28 Nov, 10:26
Financial	55	63	59	79	74.68 %	28 Nov, 10:26
average	111.3	122.2	116.8	144.2	80.98 %	

Table 10. The integers $1, \mu_1, \mu_2, \dots, \mu_c$ are partitions of $\{1, 2, \dots, m\}$ used in the models that generates the spreads in Table 1.

partition	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6
	7	13	19	25	46	125

Table 11. The calibrated parameters that gives the model spreads in Table 1 obtained via the ODE-solver and 1000-iterations.

	a	$b^{(1)}$	$b^{(2)}$	$b^{(3)}$	$b^{(4)}$	$b^{(5)}$	$b^{(6)}$	
2004/08/04	33.07	16.3	86.24	126.2	200.3	0	1379	$\times 10^{-4}$
2006/11/28	24.9	13.93	73.36	62.9	0.2604	2261	5904	$\times 10^{-4}$
2008/03/07	44.2	22.66	159.8	0	6e-008	1107	779700	$\times 10^{-4}$

Table 12. The index-CDS spread and average CDS-spread as function of the jump parameter $b^{(6)}$, when holding the other parameters in $\boldsymbol{a}$ fixed Here $\boldsymbol{a}$ is given as in Table 11. We consider the 2008-03-07 portfolio and also display the relative differences in terms of original spreads given in Table 1, i.e. when $b^{(6)} = 77.97$ (see Table 11).

	$b^{(6)} = 6.5$	$b^{(6)} = 2.5$	$b^{(6)} = 1.5$
index	143.7515	143.0021	142.3227
rel.diff (%)	0.3616	0.8810	1.3519
avg. CDS	143.3091	142.5643	141.8891
rel.diff (%)	0.3606	0.8784	1.3479

Table 13. The calibrated parameters with the Padé approach and the ODE-method as well as the relative difference in terms of the ODE-case. The calibration is done on the 2008-03-07 portfolio. In both approaches the same initial parameters where used in the calibration routine, as well as the number of iterations (100 iterations).

2008-03-07	a	$b^{(1)}$	$b^{(2)}$	$b^{(3)}$	$b^{(4)}$	$b^{(5)}$	$b^{(6)}$	
ODE	43.9	23.03	152.9	12.81	2e-008	1028	103800	$\times 10^{-4}$
Pade	43.8	23.17	148.8	23.05	0.000502	968.8	83200	$\times 10^{-4}$
rel.diff (%)	0.2286	0.5718	2.635	79.88	2323000	5.757	19.84	$\times 1$

Table 14. The model-spreads (computed with the ODE-method), the Monte Carlo spreads (with 10^6 replications) and their relative errors in terms of the model-spreads. The parameters are given by Table 11.

2004-08-04	$[0,3]$	$[3,6]$	$[6,9]$	$[9,12]$	$[12,22]$	index
ODE	27.6000	167.9997	70.0005	42.9994	20.0004	42.0185
Monte Carlo	27.6037	168.3941	70.0903	43.1534	19.9063	42.0554
rel.error (%)	0.0135	0.2348	0.1283	0.3581	0.4705	0.0878

2006-11-28	$[0,3]$	$[3,6]$	$[6,9]$	$[9,12]$	$[12,22]$	index
ODE	14.5001	62.4778	18.0727	6.8718	3.4169	26.1464
Monte Carlo	14.5668	62.7497	18.3214	6.8874	3.4108	26.1874
rel.error (%)	0.4600	0.4352	1.3761	0.2268	0.1778	0.1567

2008-03-07	$[0,3]$	$[3,6]$	$[6,9]$	$[9,12]$	$[12,22]$	index
ODE	46.5005	567.9742	369.9515	233.9651	149.9112	144.2732
Monte Carlo	46.5070	568.5201	370.6432	233.8884	149.8159	144.3121
rel.error (%)	0.0140	0.0961	0.1870	0.0328	0.0636	0.0270

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